Name: _____

- This packet is due at **12:00 pm PDT (noon) on Friday, May 21**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Find the power series for each function with the given center a, and identify its interval of convergence.

   (a) $f(x) = \dfrac{x}{1+x^2}$, $a = 0$

$$f(x) = \frac{x}{1+x^2} = x \cdot \frac{1}{1-(-x^2)} = x \sum_{n=0}^{\infty} (-x^2)^n$$

$$= x \cdot \sum_{n=0}^{\infty} (-1)^n x^{2n} = \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot x^{2n+1}}$$

converges iff $|-x^2| < 1 \iff x^2 < 1$

$\iff |x| < 1$

$\iff \boxed{x \in (-1, 1)}$

(b) $f(x) = \dfrac{x^2 - 6x + 9}{7 + x}$, $a = 3$

$$f(x) = \frac{(x-3)^2}{7+(x-3+3)} = \frac{(x-3)^2}{10-(-(x-3))}$$

$$= \frac{(x-3)^2}{10} \cdot \frac{1}{1-\left(\frac{-(x-3)}{10}\right)} = \frac{(x-3)^2}{10} \cdot \sum_{n=0}^{\infty} \left(\frac{-(x-3)}{10}\right)^n$$

$$= \frac{(x-3)^2}{10} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{10^n}$$

$$= \left| \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^{n+2}}{10^{n+1}} \right|$$

Converges iff $\left|\frac{-(x-3)}{10}\right| < 1 \iff |x-3| < 10$

$\iff \boxed{x \in (-7, 13)}$

2. Use partial fractions to find the power series of each function centered at 0. What is the interval of convergence for the power series?

(a) $\dfrac{4}{(x-3)(x+1)}$ $= \dfrac{A}{x-3} + \dfrac{B}{x+1} \rightarrow 4 = A(x+1) + B(x-3)$

$x=-1 \rightarrow 4 = B(-4)$

$\rightarrow B = -1$

$x = 3 \rightarrow 4 = A \cdot 4$

$\rightarrow A = 1$

$$\frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1} = \frac{1}{-3}\cdot\frac{1}{1-\frac{x}{3}} - \frac{1}{1-(-x)}$$

$$= \frac{1}{-3}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$

converges iff $x \in (-3,3)$

converges iff $x \in (-1,1)$

$$= \sum_{n=0}^{\infty}\frac{x^n}{-3^{n+1}} - \sum_{n=0}^{\infty}(-1)^n x^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{(-3)^{n+1}} - (-1)^n\right)x^n$$

converges iff both previous series converge, i.e. $x \in (-1,1)$

(b) $\frac{5}{(x^2+4)(x^2-1)} = \frac{A}{x^2+4} + \frac{B}{x^2-1} \to 5 = A(x^2-1) + B(x^2+4)$

$x=1 \to 5 = B\cdot 5 \to B=1$

$\Rightarrow 5 = A(x^2-1) + (x^2+4)$

$x=0 \to 5 = A(-1) + 4$

$\to 1 = -A \to A=-1$

$$\frac{5}{(x^2+4)(x^2-1)} = \frac{-1}{x^2+4} + \frac{1}{x^2-1} = \frac{1}{-4-x^2} - \frac{1}{1-x^2}$$

$$= \frac{1}{-4}\cdot\frac{1}{1-\left(\frac{x^2}{-4}\right)} - \frac{1}{1-x^2} = \frac{1}{-4}\cdot\sum_{n=0}^{\infty}\left(\frac{x^2}{-4}\right)^n - \sum_{n=0}^{\infty}(x^2)^n$$

$$= \frac{1}{-4}\cdot\sum_{n=0}^{\infty}\frac{x^{2n}}{(-4)^n} - \sum_{n=0}^{\infty}x^{2n} \quad \text{converges iff } x\in(-2,2)$$

$\sum_{n=0}^{\infty}x^{2n}$ converges iff $x\in(-1,1)$

$$= \sum_{n=0}^{\infty}\frac{x^{2n}}{(-4)^{n+1}} - \sum_{n=0}^{\infty}x^{2n} = \sum_{n=0}^{\infty}\left(\frac{1}{(-4)^{n+1}} - 1\right)x^{2n}$$

→ Converges iff both previous series converge, i.e. $x\in(-1,1)$

3. Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x \in (-1,1)$. Use term-by-term differentiation or integration to find power series for each function centered at the given point. ~~Write your conclusion as a sentence.~~ Be sure to include the interval of convergence.

(a) $\ln(1-x^2)$ at $a=0$

$$\frac{d}{dx}\left(\ln(1-x^2)\right) = \frac{1}{1-x^2}\cdot(-2x)$$

converges iff $|x^2|<1$ iff $x\in(-1,1)$

$\Rightarrow$ radius of convergence of $\sum_{n=0}^{\infty} -2x^{2n+1}$ is 1.

$$= (-2x)\cdot\sum_{n=0}^{\infty}(x^2)^n$$

$$= (-2x)\cdot\sum_{n=0}^{\infty} x^{2n}$$

$$= \sum_{n=0}^{\infty} -2x^{2n+1}$$

$$\Rightarrow \ln(1-x^2) = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2}$$

$$x=0 \to 0 = C + \sum_{n=0}^{\infty}\frac{-2\cdot 0}{2n+2} = C$$

$$\Rightarrow \boxed{\ln(1-x^2) = \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2}}$$

↳ same radius of convergence as $\sum_{n=0}^{\infty} -2x^{2n+1}$, so we only need to check $x=\pm 1$

$x=1$: $\sum_{n=0}^{\infty}\frac{-2}{2n+2} = -\sum_{n=0}^{\infty}\frac{1}{n+1} = -\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-test

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-2)(-1)^{2n+2}}{2n+2} = \sum_{n=0}^{\infty}\frac{-2}{2n+2} = \nearrow$ diverges by the p-test

$$\Rightarrow \boxed{I=(-1,1)}$$

(b) $\arctan(x^2)$ at $a = 0$

$$\frac{d}{dx}\left(\arctan(x^2)\right) = \frac{1}{1+(x^2)^2}\cdot 2x = \frac{1}{1-(-x^4)}\cdot 2x$$

$$= 2x\cdot\sum_{n=0}^{\infty}(-x^4)^n = 2x\sum_{n=0}^{\infty}(-1)^n\cdot x^{4n}$$

$$= \sum_{n=0}^{\infty} 2\cdot(-1)^n\cdot x^{4n+1}$$

radius of convergence = 1 (see previous problem)

$$\Rightarrow \arctan(x^2) = C + \sum_{n=0}^{\infty} 2\cdot(-1)^n\frac{x^{4n+2}}{4n+2}$$

$$x = 0 \to \quad 0 = C + \sum_{n=0}^{\infty}\frac{2\cdot(-1)^n}{4n+2}\cdot 0 = C$$

$$\Rightarrow \boxed{\arctan(x^2) = \sum_{n=0}^{\infty}\frac{2\cdot(-1)^n}{4n+2}\cdot x^{4n+2}}$$

As in (3a), the radius of convergence is 1, so we only need to check $x = \pm 1$

$x = 1$: $\sum_{n=0}^{\infty}\frac{2(-1)^n}{4n+2}$ converges by the A.S.T.

$x = -1$: $\sum_{n=0}^{\infty}\frac{2(-1)^n}{4n+2}\cdot(-1)^{4n+2} = \sum_{n=0}^{\infty}\frac{2(-1)^n}{4n+2}$ converges by the A.S.T.

$$\to \boxed{I = [-1, 1]}$$

4. Express $\sum_{n=0}^{\infty} \frac{1}{x^n}$ as a ratio of two polynomials.

$$\sum_{n=0}^{\infty} \frac{1}{x^n} = \sum_{n=0}^{\infty} (x^{-1})^n = \frac{1}{1-x^{-1}} \cdot \frac{x}{x} = \frac{x}{x-1}$$

when $|x^{-1}| < 1 \iff |x| > 1$

$\iff x \in (-\infty, -1) \cup (1, \infty)$

5. Let $f(x) = 2\sum_{n=0}^{\infty} x^n$ and $g(x) = \sum_{n=0}^{\infty} nx^n$. Find the power series and interval of convergence of $f(x)g(x)$.

n^{th} coefficient of $f(x)$ is $c_n = 2$

n^{th} coefficient of $g(x)$ is $d_n = n$

n^{th} coefficient of $f(x) \cdot g(x)$ is

$$c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0 = 2 \cdot n + 2(n-1) + 2(n-2) + \cdots + 2 \cdot 1 + 2 \cdot 0$$
$$= 2 \cdot \sum_{j=1}^{n} j = 2 \cdot \frac{n \cdot (n+1)}{2} = n(n+1)$$

$$\longrightarrow f(x) \cdot g(x) = \sum_{n=0}^{\infty} n(n+1) x^n$$

$$\lim_{n\to\infty}\left|\frac{(n+1)(n+2)x^{n+1}}{n(n+1)x^n}\right| = |x|\cdot\lim_{n\to\infty}\frac{n^2+3n+2}{n^2+n} = |x|$$

$$|x|<1 \iff x\in(-1,1)$$

$x=1$: $\sum_{n=0}^{\infty} n(n+1)$ diverges by the divergence test

$x=-1$: $\sum_{n=0}^{\infty} n(n+1)(-1)^n$ diverges by the divergence test

$$\Rightarrow \boxed{I=(-1,1)}$$